DSM for solving ill-conditioned linear algebraic systems [*†]

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Abstract

A standard way to solve linear algebraic systems $Au = f$, $(*)$ with ill-conditioned matrices A is to use variational regularization. This leads to solving the equation $(A^*A + aI)u = A^*f_\delta$, where a is a regularization parameter, and f_δ are noisy data, $||f - f_\delta|| \leq \delta$. Numerically it requires to calculate products of matrices A^*A and inversion of the matrix $A^*A + aI$ which is also ill-conditioned if $a > 0$ is small. We propose a new method for solving (*) stably, given noisy data f_δ. This method, the DSM (Dynamical Systems Method) is developed in this paper for selfadjoint A. It consists in solving a Cauchy problem for systems of ordinary differential equations.

1 Introduction

Consider a linear algebraic system

$$A^*Au = f, \quad f \in R(A), \tag{1}$$

where A is a linear operator in $n-$dimensional Euclidean space, $A = A^*$, $R(A)$ is the range of A, and $N := \{u : Au = 0\}$ is the null-space of A. Let $k(A) := ||A||||A^{-1}||$ denote the condition number of A. If A is singular, i.e., N is not trivial, then we set $k(A) = \infty$. Problem (1.1) is called ill-conditioned if $k(A) >> 1$. In this case small perturbations of f may lead to large perturbations of the solution u, so problem (1.1) is ill-posed. Such problems are often solved by variational regularization. This method consists in finding global minimizer of the functional $F(u) = ||Au - f_\delta||^2 + a||u||^2$, where $a = const > 0$ is a regularization parameter, and f_δ are noisy data, $||f - f_\delta|| \leq \delta$. The global minimizer of the quadratic functional F is the unique solution to the linear

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algebraic system $(A^*Au + aI)u_{a,\delta} = A^*f_\delta$, where I is the unit matrix. This system has a unique solution $u_{a,\delta} = (A^*Au + aI)^{-1}A^*f_\delta$. Calculation of the matrix A^*A requires multiplication of two matrices. For large n this is a time-consuming operation. Condition number of the matrix A^*A is $k^2(A)$, so it is much larger than $k(A)$ since $k(A) >> 1$. If a is small, then the condition number of the matrix $A^*A + aI$ is also large. Therefore, inversion of the matrix $A^*A + aI$ is numerically difficult if a is small. An additional difficulty consists in choosing the regularization parameter a as a function of δ in such a way that the element $u_\delta := u_{a(\delta),\delta}$ would converge to a solution of (1.1) as $\delta \to 0$. There are a priori and a posteriori methods for choosing such $a(\delta)$. The theory of variational regularization is presented in many books and papers (see e.g., [1], Chapter 2).

Our goal is to develop a new method for for solving ill-conditioned problems (1.1). This method, which we call the DSM (Dynamical Systems Method), does not require inversion of matrices and their multiplications. It requires solving a Cauchy problem for a system of ordinary diferential equations. The theoretical development of DSM is presented in [2],[3]. The author hopes that DSM will be an efficient numerical method for solving ill-conditioned linear algebraic systems. The assumption that A is selfadjoint, which is used for simplicity in this paper, can be relaxed. However, all generalizations will be considered elsewhere. Here we concentrate on the assumptions leading to the simplest arguments.

The idea of our method is simple: it is based on the formula

$$B^{-1}(e^{Bt} - I) = \int_0^t e^{Bs} ds, \tag{2}$$

where B is a linear boundedly invertible operator. If

$$\lim_{t\to\infty} ||e^{Bt}|| = 0, \tag{3}$$

then

$$B^{-1} = -\lim_{t\to\infty} \int_0^t e^{Bs} ds. \tag{4}$$

The integral $\int_0^t e^{B(t-s)} ds f$ is the solution to the Cauchy problem

$$\dot{u} = Bu + f, \quad u(0) = 0, \quad \dot{u} = \frac{du}{dt}. \tag{5}$$

Therefore, one can calculate the inverse of an operator satisfying condition (3) by solving a Cauchy problem. We want to calculate $A^{-1}f$. Let us take $B = i(A + iaI)$, where I is the identity operator and $a > 0$ is a parameter which we take to zero later. Condition (3) is satisfied if $A = A^*$ and $a > 0$ because under these assumptions $||e^{iAt}|| = 1$ and $\lim_{t\to\infty} e^{-at} = 0$. We write $A + ia$ in place of $A + iaI$ below.

Consider the problem:

$$\dot{u}_a = i(A + ia)u_a + f, \quad u_a(0) = 0. \tag{6}$$

Its unique solution is $u_a = \int_0^t e^{i(A+ia)(t-s)} ds f$. Our results are the following two theorems.

Theorem 1. *One has*

$$-i \lim_{a\to 0} \lim_{t\to\infty} u_a(t) = y, \tag{7}$$

where y is the minimal-norm solution to (1.1).

Note that if

$$\dot{v}_a = i(A+ia)v_a - if, \quad v_a(0) = 0,$$

then $\lim_{a\to 0} \lim_{t\to\infty} v_a(t) = y$.

If f_δ is given in place of f, $||f - f_\delta|| \le \delta$, then one solves the problem:

$$\dot{u}_{a,\delta} = i(A+ia)u_{a,\delta} + f_\delta, \quad u_{a,\delta}(0) = 0. \tag{8}$$

Theorem 2. *If t_δ and $a = a(\delta)$ are such that*

$$\lim_{\delta\to 0} t_\delta = \infty, \quad \lim_{\delta\to 0} a(\delta) = 0, \quad \lim_{\delta\to 0} \frac{\delta}{a(\delta)} = 0, \quad \lim_{\delta\to 0} a(\delta)t_\delta = \infty, \tag{9}$$

then

$$\lim_{\delta\to 0} ||u_\delta - iy|| = 0, \tag{10}$$

where $u_\delta := u_{a(\delta),\delta}$.

In the next Section proofs are given.

2 Proofs

Proof of Theorem 1. By the argument in Section 1, we have

$$u_a(t) = [-i(A+ia)]^{-1}(I - e^{i(A+ia)t})f, \tag{11}$$

and $\lim_{t\to\infty} ||e^{i(A+ia)t}|| = 0$. Thus, $\lim_{t\to\infty} u_a(t) = i(A+ia)^{-1}f$. Consequently,

$$-i \lim_{a\to 0} i(A+ia)^{-1}f = \lim_{a\to 0}(A+ia)^{-1}Ay = y,$$

as claimed in Theorem 1. Let us explain the last step. Using the spectral theorem for the selfadjoint operator A, one gets

$$\lim_{a\to 0} ||(A+ia)^{-1}Ay - y||^2 = \lim_{a\to 0} \int_{\mathbb{R}} |s(s+ia)^{-1} - 1|^2 d(E_s y, y) = \lim_{a\to 0} \int_{\mathbb{R}} \frac{a^2}{a^2+s^2} d(E_s y, y) = 0. \tag{12}$$

Here we have used the assumption $y \perp N$, which implies that $\lim_{0<b\to 0} \int_{-b}^0 d(E_s y, y) = 0$. This relation allows one to pass to the limit $a \to 0$ under the integral sign in (12). Theorem 1 is proved. □

Remark 1. In our case the operator A is bounded, so the integration in (12) is taken over a finite interval $[-||A||, ||A||]$. Moreover, in a finite-dimensional space the spectrum

of the operator A consists of finitely many eigenvalues, and the integral in (12) reduces to a finite sum. Our proof of Theorem 1 allows one to use it in infinite-dimensional spaces.

Proof of Theorem 2. One has

$$||u_\delta - y|| \leq ||u_{a(\delta)} - y|| + ||\int_0^{t_\delta} e^{i(A+ia)(t-s)} ds (f_\delta - f)||. \tag{13}$$

The argument given in the proof of Theorem 1 shows that

$$\lim_{\delta \to 0} ||u_{a(\delta)}(t_\delta) - y|| = 0 \tag{14}$$

provided that

$$\lim_{\delta \to 0} a(\delta) = 0, \ \lim_{\delta \to 0} t_\delta = \infty, \ \text{and} \ \lim_{\delta \to 0} a(\delta) t_\delta = \infty.$$

One estimates the integral

$$J := ||\int_0^{t_\delta} e^{i(A+ia)(t-s)} ds (f_\delta - f)|| \leq \frac{\delta}{a}. \tag{15}$$

Thus

$$||u_\delta - iy|| \leq \frac{\delta}{a} + o(1), \tag{16}$$

where the term $o(1)$ comes from (14). Therefore, if $\lim_{\delta \to 0} \frac{\delta}{a(\delta)} = 0$ then $\lim_{\delta \to 0} ||u_\delta - y|| = 0$, and Theorem 2 is proved. □

Remark 2. The assumption $A = A^*$ allows one to give a short and simple proof of Theorems 1,2. However, this assumption can be replaced by more general assumptions. For example, one can assume that A has Jordan chains of length one only. In other words, that the resolvent of A has only simple poles. Under this more general assumption the Cauchy problem we have used should be also modified, in general. If $A = A^*$, then our proofs of theorems 1 and 2 remain valid for operator A in a Hilbert space, and not only in a finite-dimensional space.

References

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